Series Solution of the Problem of Two Fixed Centers

R. K. Choudkhari

Let the x axis be directed along the line joining the two centers. The origin of coordinates is placed at the midpoint between the centers. The unit of distance is taken as half the distance between the centers and the unit of mass as the sum of the masses of both centers. Furthermore, let the mass of the center c_2 be m, and, therefore, that of the other center is $1 - m$. Also, let r_1 be the distance from the center c_1 to a point M and r_2 the distance from c_2 to the same point.

Consequently we have

$$r_1^2 = (x-1)^2 + y^2 + z^2$$
$$r_2^2 = (x+1)^2 + y^2 + z^2 \qquad (1)$$

The differential equations of motion are given by

$$\frac{d^2x}{dt^2} = -k^2(1-m)\frac{x-1}{r_1^3} - k^2m\frac{x+1}{r_2^3}$$
$$\frac{d^2y}{dt^2} = -k^2(1-m)\frac{y}{r_1^3} - k^2m\frac{y}{r_2^3} \qquad (2)$$
$$\frac{d^2z}{dt^2} = -k^2(1-m)\frac{z}{r_1^3} - k^2m\frac{z}{r_2^3}$$

Let us introduce auxiliary variables

$$\rho_i = r_i^2 \qquad \sigma_i = r_i^{-3} \qquad (i = 1,2) \qquad (3)$$

so that

$$2\rho_i\frac{d\sigma_i}{dt} + 3\sigma_i\frac{d\rho_i}{dt} = 0 \qquad (4)$$

$$\rho_1 = (x-1)^2 + y^2 + z^2$$
$$\rho_2 = (x+1)^2 + y^2 + z^2 \qquad (5)$$

and the differential equations of motion become

$$d^2x/dt^2 = -k^2(1-m)(x-1)\sigma_1 - k^2m(1+x)\sigma_2$$
$$d^2y/dt^2 = -k^2(1-m)y\sigma_1 - k^2my\sigma_2 \qquad (6)$$
$$d^2z/dt^2 = -k^2(1-m)z\sigma_1 - k^2mz\sigma_2$$

System of equations (6) has two integrals: vis viva integral

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{2k^2(1-m)}{r_1} + \frac{2k^2m}{r_2} + h \qquad (7)$$

and the integral of areas

$$y\dot{z} - z\dot{y} = a$$

There are seven equations (4), (5), (6) in seven unknowns $x, y, z, \rho_1, \rho_2, \sigma_1, \sigma_2$.

Assume that the solution of these equations can be represented by infinite series

$$x = \sum_{\nu=0}^{\infty}\alpha_\nu t^\nu \qquad y = \sum_{\nu=0}^{\infty}\beta_\nu t^\nu \qquad z = \sum_{\nu=0}^{\infty}z_\nu t^\nu$$
$$\rho_1 = \sum_{\nu=0}^{\infty}a_\nu t^\nu \qquad \rho_2 = \sum_{\nu=0}^{\infty}b_\nu t^\nu \qquad (8)$$
$$\sigma_1 = \sum_{\nu=0}^{\infty}c_\nu t^\nu \qquad \sigma_2 = \sum_{\nu=0}^{\infty}d_\nu t^\nu$$

Translated from Vestnik Moskovskogo Universiteta, Astronomiia (Bulletin of Moscow University, Astronomy), no. 4, 83–91 (1962). Translated by Igor Jurkevich.

The vis viva integral (7) will not be used, since it will be found useful as a check in computing position as a function of time.

Define

$$(n+2)^{(2)} = (n+1)(n+2)$$
$$(\alpha c)_n = \sum_{\nu=0}^{n}\alpha_\nu c_{n-\nu} \qquad (9)$$

Substituting expressions (8) in Eqs. (6) we obtain

$$(n+2)^{(2)}\alpha_{n+2} = -k^2(1-m)(\alpha c)_n - k^2m(\alpha d)_n + k^2(1-m)c_n - k^2md_n$$
$$(n+2)^{(2)}\beta_{n+2} = -k^2(1-m)(\beta c)_n - k^2m(\beta d)_n \qquad (10)$$
$$(n+2)^{(2)}r_{n+2} = -k^2(1-m)(rc)_n - k^2m(rd)_n$$

From Eq. (5) it follows that

$$a_n = (\alpha\alpha)_n + (\beta\beta)_n + (rr)_n - 2\alpha_n + \epsilon_n$$
$$b_n = (\alpha\alpha)_n + (\beta\beta)_n + (rr)_n + 2\alpha_n + \epsilon_n \qquad (11)$$

where $\epsilon n = 1$ for $n = 0$, and $\epsilon n = 0$ for $n \neq 0$, and

$$-2na_0c_n = \sum_{\nu=0}^{n-1}(3n-\nu)c_\nu a_{n-\nu}$$
$$-2nb_0d_n = \sum_{\nu=0}^{n-1}(3n-\nu)d_\nu b_{n-\nu} \qquad (12)$$

Six constants of integration are obtained from the initial values of $x, y, z, \dot{x}, \dot{y}, \dot{z}$. Thus at $t = 0$ we take $(\alpha_0, \alpha_1, \beta_0, \beta_1, r_0, r_1)$. From expressions (11) and (12) we have

$$a_0 = (\alpha_0 - 1)^2 + \beta_0^2 + r_0^2$$
$$b_0 = (\alpha_0 + 1)^2 + \beta_0^2 + r_0^2$$
$$c_0 = a_0^{-3/2} \qquad d = b_0^{-3/2}$$
$$a_1 = 2(\alpha_0\alpha_1 + \beta_0\beta_1 + r_0r_1) - 2\alpha_1$$
$$b_1 = 2(\alpha_0\alpha_1 + \beta_0\beta_1 + r_0r_1) - 2\alpha_1$$
$$-2a_0c_1 = 3a_1a_0^{-3/2}$$
$$-2b_0d_1 = 3b_1b_0^{-3/2}$$

Subsequently we evaluate α_2, β_2, r_2. Substituting quantities a_1, b_1, c_1, d_1 we obtain

$$1.2\ \alpha_2 = k^2(1-m)a_0^{-3/2}(1-\alpha_0) - k^2mb_0^{-3/2}(1+\alpha_0)$$
$$1.2\ \beta_2 = k^2(1-m)\beta_0a_0^{-3/2} - k^2m\beta_0b_0^{-3/2}$$
$$1.2\ r_2 = -k^2(1-m)r_0a_0^{-3/2} - k^2mr_0b_0^{-3/2}$$
$$a_2 = \alpha_1^2 + \beta_1^2 + r_1^2 - k^2mb_0^{-3/2}\{(\alpha_0^2 - 1) + \beta_0^2 + r_0^2\} - k^2(1-m)a_0^{1/2}$$

The values $\alpha_3, \beta_3, r_3, a_3, b_3, c_3, d_3$ are obtained in an analogous way. Thus the series can be written as

$$x = \alpha_0 + \alpha_1t - \frac{k^2t^2}{2!}[(1-m)(\alpha_0-1)a_0^{-3/2} + m(\alpha_0+1)b_0^{-3/2}] + \alpha_3t^3 + \ldots$$

$$y = \beta_0 + \beta_1t - \frac{k^2t^2}{2!}\beta_0[(1-m)a_0^{-3/2} + mb_0^{-3/2}] + \beta_3t^3 + \ldots$$

$$z = r_0 + r_1t - \frac{k^2t^2}{2!}r_0[(1-m)a_0^{-3/2} + mb_0^{-3/2}] + r_3t^3 + \ldots$$

$$\rho_1 = (\alpha_0 - 1)^2 + \beta_0^2 + r_0^2 + 2t\{\alpha_1(\alpha_0 - 1) + \beta_0\beta_1 + r_0r_1\} + t^2[\alpha_1^2 + \beta_1^2 + r_1^2 - k^2mb_0^{-3/2}\{(\alpha_0^2 - 1) + \beta_0^2 + r_0^2\} - k^2(1-m)a_0^{-1/2}] + a_3t^3 + \ldots \quad \rho_2 = (\alpha_0 + 1)^2 + \beta_0^2 + r_0^2 + 2t\{\alpha_1(\alpha_0 + 1) + \beta_0\beta_1 + r_0r_1\} + t^2[\alpha_1^2 + \beta_1^2 + r_1^2 - k^2(1-m)a_0^{-3/2}\{(\alpha_0^2 - 1) + \beta_0^2 + r_0^2\} - k^2mb_0^{-1/2}] + b_3t^3 + \ldots$$

$$\sigma_1 = a_0^{-3/2} - 3a_0^{-5/2}\{\alpha_1(\alpha_0 - 1) + \beta_0\beta_1 + r_0r_1\}t - \tfrac{3}{2}a_0^{-5/2}[\alpha_1^2 + \beta_1^2 + r_1^2 - k^2m\{(\alpha_0^2 - 1) + \beta_0^2 + r_0^2\}b_0^{-3/2} - k^2(1-m)a_0^{-1/2}]t^2 + \tfrac{15}{2}a_0^{-7/2}[\alpha_1(\alpha_0 - 1) + \beta_0\beta_1 + r_0r_1]^2t^2 + c_3t^3 + \ldots$$

$$\sigma_2 = b_0^{-3/2} - 3b_0^{-5/2}\{\alpha_1(\alpha_0 + 1) + \beta_1\beta_0 + r_1r_0\}t + d_2t^2 + \ldots$$

To use the series solution (8), its convergence must be ascertained.

Defining

$$H_\nu = \frac{\lambda^\nu}{(\nu + 2)^{(2)}} \qquad (\lambda > 0) \tag{13}$$

$$s_n = \sum_{\nu=1}^{n} \frac{1}{\nu} \tag{14}$$

we can write an identity which Steffenson[1] gave as

$$\sum_{\nu=0}^{n} H_\nu H_{n-\nu} = 2\lambda^n \frac{2s_{n-1} + n + 1}{(n + 4)^{(3)}} \tag{15}$$

$$\sum_{\nu=0}^{n-1} H_\nu H_{n-\nu} = \lambda^n \frac{4s_{n-1} + \frac{3}{2}n - 1 - \frac{3}{(n+1)}}{(n + 4)^{(3)}} \quad (n \geqslant 1) \tag{16}$$

$$\sum_{\nu=2}^{n-2} H_\nu H_{n-\nu} = \frac{2}{3}\lambda^n \frac{6s_n + n - 10 - \frac{12}{n}}{(n + 4)^{(3)}} \quad (n \geqslant 4) \tag{17}$$

since

$$\lambda^{-n}H_\nu H_{n-\nu} = \left(\frac{1}{\nu + 1} + \frac{1}{n - \nu + 1}\right)\frac{1}{(n + 3)^{(2)}} - \left(\frac{1}{\nu + 2} + \frac{1}{n - \nu + 2}\right)\frac{1}{(n + 4)^{(2)}} \tag{18}$$

Furthermore from the identity

$$\lambda^{-n}\nu H_\nu H_{n-\nu} = \left(\frac{n + 1}{n - \nu + 1} - \frac{1}{\nu + 1}\right)\frac{1}{(n + 3)^{(2)}} - \left(\frac{n + 2}{n - \nu + 2} - \frac{2}{\nu + 2}\right)\frac{1}{(n + 4)^{(2)}} \tag{19}$$

we obtain

$$\sum_{\nu=1}^{n-1} \nu H_\nu H_{n-\nu} = \frac{n\lambda^n}{(n + 4)^{(3)}}\left(2s_{n+1} + \frac{n}{2} - 2 - \frac{3}{n + 1}\right) \quad (n \geqslant 2) \tag{20}$$

Assuming $n \geqslant 3$ and separating constants of integration, expression (11) can be written as

$$a_n = 2(\alpha_0\alpha_n + \beta_0\beta_n + r_0r_n + \alpha_1\alpha_{n-1} + \beta_1\beta_{n-1} + r_1r_{n-1}) + \sum_{\nu=2}^{n-2} (\alpha_\nu\alpha_{n-\nu} + \beta_\nu\beta_{n-\nu} + r_\nu r_{n-\nu})^{-2a_n} \tag{21}$$

Note that the sum vanishes for $n = 3$.

Now it is assumed that for $n \geqslant 3$ and $2 \leqslant \nu \leqslant n$ the following inequalities hold

$$|\alpha_\nu| \leqslant \alpha H_\nu \qquad |\beta_\nu| \leqslant \beta H_\nu \qquad |r_\nu| \leqslant rH_\nu \tag{22}$$

where α, β, and r are positive constants. In this case expression (21) yields

$$a_n| \leqslant 2(\alpha|\alpha_0| + \beta|\beta_0| + r|r_0|)H_n + 2(\alpha|\alpha_1| + \beta|\beta_1| + r|r_1|)H_{n-1} + 2\alpha H_n + (\alpha^2 + \beta^2 + r^2)\sum_{\nu=2}^{n-2} H_\nu H_{n-\nu} \tag{23}$$

Employing (17) we find

$$|a_n| \leqslant 2[\alpha(|\alpha_0| + 1) + \beta|\beta_0| + r|r_0)]\frac{\lambda^n}{(n + 2)^{(2)}} + 2(\alpha|\alpha_1| + |\beta_1| + r|r_1|)\frac{\lambda^{n-1}}{(n + 1)^{(2)}} + \frac{2}{3}(\alpha^2 + \beta^2 + r^2)\left(6s_n + n - 10 - \frac{12}{n}\right)\frac{\lambda^n}{(n + 4)^{(3)}} \tag{24}$$

In this expression the last term is ignored for $n \leqslant 3$, and consequently (24) holds for $n \geqslant 3$. A sufficient condition for $|a_n| < AH_n$ to hold can be written as

$$\leqslant A\frac{\lambda^n}{(n + 2)^{(2)}}$$

where A is a given positive number.

Thus, we have

$$\alpha(|\alpha_0| + 1) + \beta|\beta_0| + r|r_0| + (\alpha|\alpha_1| + \beta|\beta_1| + r|r_1|) \times \frac{n + 2}{n\lambda} + \frac{1}{3}(\alpha^2 + \beta^2 + r^2)\left(6s_n + n - 10 - \frac{12}{n}\right) \times \frac{n + 1}{(n + 4)^{(2)}} \leqslant \frac{A}{2} \tag{25}$$

It is convenient to express this condition in a form independent of n. Let us write

$$\frac{n + 2}{n} = 1 + \frac{2}{n} \leqslant \frac{5}{3} \qquad (n \geqslant 3) \tag{26}$$

Evidently we have

$$\left(6s_n + n - 10 - \frac{12}{n}\right)\quad \frac{n + 1}{(n + 4)^{(2)}} \leqslant 2$$

For $n = 3$ and $n = 4$ it is assumed that $n \geqslant 5$. Thus

$$s_n \leqslant s_k + \frac{n - k}{k + 1} \qquad (n \geqslant k) \tag{27}$$

In particular

$$s_n \leqslant s_5 + \frac{n - 5}{6} = \frac{n}{6} + \frac{29}{20} \qquad (n \geqslant 5)$$

Substituting this inequality in $6S_n + n - 10 - 12/n$ we have

$$\frac{2n - 1.3 - (12/n)}{n + 4}\,\frac{n + 1}{n + 3} < 2 \tag{28}$$

From Eq. (25) we have

$$\alpha(|\alpha_0| + 1) + \beta|\beta_0| + r|r_0| + \frac{5}{3\lambda}(\alpha|\alpha_1| + \beta|\beta_1| + r|r_1|) + \frac{2}{3}(\alpha^2 + \beta^2 + r^2) \leqslant \frac{A}{2} \tag{29}$$

which is the condition that $|a_n| \leqslant AH_n$ for $n \geqslant 3$. Comparing the two results for (11) we find that a sufficient condition for $|b_n| \leqslant BH_n$ is the same as that given by (29), namely

$$\alpha(|\alpha_0| + 1) + \beta|\beta_0| + r|r_0| + \frac{5}{3\lambda}(\alpha|\alpha_1| + \beta|\beta_1| + r|r_1|) + \frac{2}{3}(\alpha^2 + \beta^2 + r^2) \leqslant \frac{B}{2} \leqslant \frac{A}{2} \tag{30}$$

if max $(A,B) = A$.

Consider now Eq. (12). Assume that for a given $n \geqslant 2$ and $1 \leqslant \nu \leqslant n$ it has been proved that

$$|a_\nu| \leqslant AH_\nu \qquad |b_\nu| \leqslant AH_\nu \tag{31}$$

and for $0 \leqslant \nu \leqslant n - 1$:

$$|c_\nu| \leqslant CH_\nu \qquad |d_\nu| \leqslant DH_\nu^2 \tag{32}$$

The first equation (12) yields

$$2na_0|c_n| \leqslant CA\left(3n\sum_{\nu=0}^{n-1}H_\nu H_{n-\nu} + \sum_{\nu=1}^{n-1}\nu H_\nu H_{n-\nu}\right)$$

for example

$$2na_0|c_n| \leqslant CA\left[3n\lambda^n\frac{4s_{n+1} + \frac{3}{2}n - 1 - \frac{3}{n+1}}{(n+4)^{(3)}} + n\lambda^n\frac{2s_{n+1} + \frac{n}{2} - 2 - \frac{3}{n+1}}{(n+4)^{(3)}}\right] = CA\left[n\lambda^n\frac{14s_{n+1} + 5n - 5 - \frac{12}{n+1}}{(n+4)^{(3)}}\right]$$

and

$$2a_0|C_n| \leqslant CA\left(14s_{n+1} + 5n - 5 - \frac{12}{n+1}\right)\frac{\lambda^n}{(n+4)^{(3)}}$$

Therefore a sufficient condition for $|c_n| \leqslant CH_n$ can be written as

$$2a_0|c_n| \leqslant CA\left(14s_{n+1} + 5n - 5 - \frac{12}{n+1}\right) \times \frac{\lambda^n}{(n+4)^{(3)}} \leqslant 2a_0C\frac{\lambda^n}{(n+2)^{(2)}}$$

for example

$$\left(14s_{n+1} + 5n - 5 - \frac{12}{n+1}\right)\frac{n+1}{(n+4)^{(2)}} \leqslant \frac{2a_0}{A}$$

Let us show that this condition can be expressed as

$$3A \leqslant a_0 \tag{33}$$

which is independent of n. Therefore, we must prove that

$$\left(14s_{n+1} + 5n - 5 - \frac{12}{n+1}\right)\frac{n+1}{(n+4)^{(2)}} \leqslant 6$$

or

$$s_{n+1} \leqslant \frac{n-1}{14} + 3 + \frac{24}{7(n+1)} \tag{34}$$

Using Glover's tables[2] we find that (34) remains valid for $n < 12$. For $n \geqslant 12$ we shall employ

$$s_{n+1} \leqslant s_{13} + [(n - 12)/14]$$

Substituting this expression in (34) and simplifying, we have

$$s_{13} < \frac{53}{14} + \frac{24}{7(n+1)}$$

It is easy to show that the inequality is satisfied identically, and, consequently, that (33) and (34) are also satisfied. Analogously, it can be shown that the sufficient condition for $|d_n| \leqslant DH_\nu$ is

$$3A \leqslant b_0 \tag{35}$$

Finally, separating the constants of integration, we shall write the first equation of (10) ($n \geqslant 2$) in the form

$$(n+2)^{(2)}\alpha_{n+2} = -k^2(1-m)(\alpha_0c_n + \alpha_1c_{n-1}) - k^2m(\alpha_0d_n + \alpha_1d_{n-1}) - k^2(1-m)\sum_{\nu=2}^{n}\alpha_\nu c_{n-\nu} - k^2m\sum_{\nu=2}^{n}\alpha_\nu d_{n-\nu} + k^2(1-m)c_n - k^2md_n \tag{36}$$

$$(n+2)^{(2)}|\alpha_{n+2}| \leqslant k^2\{|\alpha_0|(1-m)C + |\alpha_0|mD + (1-m)C + mD\}H_n + k^2\{(1-m)|\alpha_1|C + m|\alpha_1|D\}H_{n-1} + k^2(1-m)\alpha C\sum_{\nu=2}^{n}H_\nu H_{n-\nu} + k^2m\alpha D\sum_{\nu=2}^{n}H_\nu H_{n-}$$

For brevity, let

$$P = \xi(1-m)C + mD\xi k^2$$

then

$$(n+2)^{(2)}|\alpha_{n+2}| \leqslant P(|\alpha_0| + 1)H_n + P|\alpha_1|H_{n-1} + \alpha P\sum_{\nu=2}^{n}H_\nu H_{n-\nu}$$

that is

$$(n+2)^{(2)}|\alpha_{n+2}| \leqslant P(|\alpha_0| + 1)\frac{\lambda^n}{(n+2)^{(2)}} + P|\alpha_1|\frac{\lambda^{n-1}}{(n+1)^{(2)}} + \alpha P\left(4s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) + \frac{\lambda^n}{(n+4)^{(3)}} \tag{37}$$

In order $|\alpha_{n+2}| \leqslant \alpha H_{n+2}$, the right-hand side of (37) must be $\leqslant (n+2)^{(2)}\alpha H_{n+2}$. Therefore, multiplying by $\lambda^{-n}[(n+4)^{(2)}/(n+2)^{(2)}]$ we have

$$P(|\alpha_0| + 1)\frac{(n+3)(n+4)}{(n+1)^2(n+2)^2} + P|\alpha_1|\frac{(n+3)(n+4)\lambda^{-1}}{(n+2)(n+1)^2n} + \frac{\alpha P}{(n+1)(n+2)^2}\left(4s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) \leqslant \alpha\lambda^2 \tag{38}$$

Obviously, for $n \geqslant 2$ we have

$$\frac{(n+3)(n+4)}{(n+1)^2(n+2)^2} = \left(1 + \frac{2}{n+2}\right)\left(1 + \frac{1}{n+2}\right) + \frac{1}{(n+1)^2} \leqslant \frac{5}{24} \tag{39}$$

$$\frac{(n+3)(n+4)}{n(n+1)^2(n+2)} = \left(1 + \frac{2}{n+2}\right)\left(1 + \frac{3}{n}\right) + \frac{1}{(n+1)^2} \leqslant \frac{5}{12} \tag{40}$$

Finally, it can be seen that

$$\frac{1}{(n+2)^2(n+1)}\left(4s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) \leqslant \frac{5}{48} \tag{41}$$

and therefore the condition (38) is taken in the form

$$P(|\alpha_0| + 1)\frac{5}{24} + P|\alpha_1|\frac{5}{12}\lambda^{-1} + \alpha P\frac{5}{48} \leqslant \alpha\lambda^2$$

or

$$1 + |\alpha_0| + \frac{4}{\lambda}|\alpha_1| + \alpha \leqslant \frac{48\alpha}{5P}\lambda^2 \tag{42}$$

Sufficient conditions for the remaining equations (10) can be written as

$$|\beta_0| + \frac{4}{\lambda}|\beta_1| + \beta \leqslant \frac{48\beta}{5p}\lambda^2 \tag{43}$$

$$|r_0| + \frac{4}{\lambda}|r_1| + r \leqslant \frac{48r}{5p}\lambda^2 \tag{44}$$

Let us summarize. If expression (22) is satisfied for $2 \leqslant \nu \leqslant 3$, expression (31) for $1 \leqslant \nu \leqslant 2$, and (32) for $0 \leqslant \nu \leqslant 2$, and, in addition, inequalities (29), (30), (33), (35), (42–44) have been satisfied, then all series of system (8) converge if the series $\Sigma H_\nu t^\nu$ converges, that is, if $|t| \leqslant 1/\lambda$.

A question now arises whether it is always possible to select λ,α,β,r, A,B,C,D for the initial conditions $\alpha_0,\alpha_1,\beta_0$, β_1,r_0,r_1, such that all the inequalities discussed previously will be satisfied. This question can be answered in the affirmative. The quantity λ can always be chosen so large that expressions (42–44) hold. Expressions (29) and (30) can be written as

$$\alpha(|\alpha_0| + 1) + \beta|\beta_0| + r|r_0| + \frac{2}{3}(\alpha^2 + \beta^2 + r^2) \leqslant \frac{A}{2} \tag{45}$$

Expressions (33) and (35) do not change. Thus it is possible to select A sufficiently small in order to satisfy (33) and (35). Following this, α,β, and r can be chosen sufficiently small to satisfy (45). Let

$$m_1 = 1 - \frac{1}{329{,}390} \qquad m_2 = \frac{1}{329{,}390}$$

and the initial conditions

$$\begin{array}{lll} \alpha_0 = 0.4 & \beta_0 = 0 & r_0 = 0 \\ \alpha_1 = 0.015 & \beta_1 = -0.01 & r_1 = -0.01 \end{array}$$

In this example the unit of distance is taken as the distance between the centers, so that in (42) we obtain $|\alpha_0| + \frac{1}{2}$ rather than $|\alpha_0| + 1$. The remaining conditions are not changed.

It can be seen that all sufficient conditions are satisfied if we choose

$$\begin{array}{lll} \lambda = \frac{1}{25} & \alpha = 0.001 & \beta = 0.002 \\ r = 0.003 & A = 0.01 & \\ C = 0.018 & D = 0.15 & \end{array}$$

The series (8) converge at least for $|t| \leqslant 25$, $k^2 = 0.0003$. The unit of time is one day.

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[2] Glover, T. W., Ann Arbor, Mich., p. 456 (1923).

Reviewer's Comment

Considerable effort has been expended in the hope that the two-fixed-center solution might lead to an understanding of the restricted three-body problem. The references listed here represent only a partial compilation of this effort.

The complete closed form solution of the two-fixed-center problem is contained in Ref. 5.

The Russian effort contained in this article represents a power series solution and, consequently, is not as complete. Moreover, the application of the two-fixed-center solution to lunar trajectories has proved rather disappointing. In particular, the trajectories describing the motion beyond the point of lunar passage is quite unrealistic. In addition, the solution of the restricted three-body problem in the vicinity of the two singularities (near the earth and near the moon) is represented by the Kepler conic section, with greater accuracy than is obtainable through the two-fixed-center approximation. An evaluation of this comparison is contained in Ref. 7.

—SAMUEL PINES
Analytical Mechanics Associates, Inc.

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[4] Wintner, A., *Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, 1947), pp. 145–147.

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[6] Davidson, M. C. and Schulz-Arenstorff, R., "Reduction of the restricted problem of three bodies to a perturbation of Euler's problem of two fixed centers," NASA Rept. MTP-AERO-61-62 (1961).

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